Copula Entropy

Theory and Applications

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Contents

Lecture 2: Theory of Copula Entropy (2)

Physical Meaning of Copula Entropy

- **Laws of Thermodynamics**
  - First law: Conservation of energy
    Energy cannot be created or destroyed and only changes from one form to another.
    $$\Delta U = Q - W \tag{1}$$
  - Second law: Entropy increase
    The total disorder of an isolated system always goes up over time.
    $$\delta Q = TdS \tag{2}$$
  - Third law: Absolute zero limit
    As temperature approaches absolute zero (0 K), a system's entropy approaches a constant minimum.

Physical Meaning of Copula Entropy

- **Physical meaning of copula entropy**
  entropy reduction due to increasing order of thermodynamic systems

$$H_c(\mathbf{x}) \leq 0 \tag{3}$$

Physical Meaning of Copula Entropy

- **Systems that keep order**
  - Earth
    constantly receiving energy from Sun to keep earth system evolving
  - Society
    receiving energy from natural environment to keep society running
  - Life
    getting food and energy from outside to keep alive

Copula Entropy: Estimation

- **Parametric** Estimation by Definition

$$H_c(\mathbf{x}) = -E(\log c(\mathbf{u}; \alpha)). \tag{4}$$

1. estimating empirical copula density $\mathbf{u}$ based on rank statistic

$$u_i = F_i(x_i) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(x_t^i \leq x_i) \tag{5}$$

2. estimating the parameter α of copula density function $c(\cdot)$ with maximum likelihood method
3. calculating according to (4)

Copula Entropy: Estimation

- **Non-Parametric** Estimation Method[1]
  1. estimating empirical copula density $\mathbf{u}$ based on rank statistic

$$u_i = F_i(x_i) = \frac{1}{T}\sum_{t=1}^{T} \mathbf{1}(x_t^i \leq x_i) \tag{6}$$

  2. estimating copula entropy with the kNN entropy estimation method
- Advantages
  - distribution-free, non-parametric
  - tuning-free, insensitive to parameters
  - good convergence
  - easy to implement
  - low computation burden

[1] Jian Ma and Zengqi Sun. "Mutual Information Is Copula Entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845, 2008, pp. 51–54.

Copula Entropy: Estimation

- **Conditional Mutual Information** Estimation

can be done with copula entropy estimators according to (7).

$$I(x; y|z) = H_c(x, z) + H_c(y, z) - H_c(x, y, z). \quad (7)$$

1. estimating the CE terms in (7) with CE estimator
2. calculating the result

Theory Growth

Classical Information Theory

- CE based MI estimation, such as GCMI[2], MICE[3], Vector Copula based MI estimation[4], Vine copula based MI and CMI estimation[5], semi-parametric MI estimation[6], kernel based CE estimation[7], non-parametric CE estimation[8]

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[3] Adrien Cortes. "Accélération des techniques d' analyse de facteurs par méthodes d' apprentissage statistique et application à un GSE". Mémoire. Institut des Sciences Financiéres et d' Assurance, 2023.

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[8] Baby A.K. and Rajesh G. "Nonparametric Estimation of Copula Entropy for Left Truncated Right Censored Data". In: *International Journal of Statistics and Reliability Engineering* 13.1 (2026), pp. 10–17.

Theory Growth

Classical Information Theory

- asymmetric MI estimation[9], CE^2[10], Bayesian dependence measure[11], and tail dependence measure[12];

[9] Soumik Purkayastha and Peter X.K. Song. "Asymmetric predictability in causal discovery: an information theoretic approach". In: *arXiv preprint arXiv:2210.14455* (2022).

[10] Lin Liu, Cong Hu, and Xiao-Jun Wu. "CE^2: A Copula Entropic Mutual Information Estimator for Enhancing Adversarial Robustness". In: *Pattern Recognition and Computer Vision*. Springer Singapore, 2024, pp. 163–174.

[11] Guillaume Marrelec and Alain Giron. "An Inferential Measure of Dependence Between Two Systems Using Bayesian Model Comparison". In: *IEEE Transactions on Systems, Man, and Cybernetics: Systems* 55.3 (2025), pp. 1671–1683. DOI: 10.1109/TSMC.2024.3509025.

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Theory Growth

Classical Information Theory

- CE based information decomposition[13];
- CE based maximal entropy[14] and maximal Tsallis CE[15].

[13] Aleksander Wieczorek and Volker Roth. "Causal Compression". In: *arXiv preprint arXiv:1611.00261* (2016), Dinu Johannes Kaufmann. "Semi-parametric Gaussian Copula Models for Machine Learning". PhD thesis. University of Basel, 2017.

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Graph Theory

- MI based graph similarity[16].

[16] Francisco Escolano et al. "The mutual information between graphs". In: *Pattern Recognition Letters* 87 (2017), pp. 12–19. DOI: https://doi.org/10.1016/j.patrec.2016.07.012. URL: https://www.sciencedirect.com/science/article/pii/S016786551630174X, Manuel Curado Navarro. "Structural Similarity: Applications to Object Recognition and Clustering". PhD thesis. Universitat d'Alacant, 2018.

Theory Growth

Information Bottleneck

- CE based information bottleneck computation[17].

[17] Mélanie Rey and Volker Roth. "Meta-Gaussian information bottleneck". In: *Advances in Neural Information Processing Systems* 25 (2012), Mélanie Rey. "Copula Models in Machine Learning". PhD thesis. University of Basel, 2015.

Theory Growth

Partial Information Decomposition

- unique information estimation[18], synergy estimation[19], and Ω information estimation[20].

[18] Ari Pakman et al. "Estimating the Unique Information of Continuous Variables in Recurrent Networks". In: *Advances in Neural Information Processing Systems* 34 (2021), pp. 20295–20307.

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Theory Growth

Information Geometry

- copula variational Bayes algorithm[21].

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Theory Growth

Copula Theory

- copula function estimation[22], Vine copula model selection[23] and circular-linear copulas[24].

[22] Longxia Qian et al. "A New Estimation Method for Copula Parameters for Multivariate Hydrological Frequency Analysis With Small Sample Sizes". In: *Water Resources Management* 36.4 (2022), pp. 1141–1157. ISSN: 1573-1650. URL: https://doi.org/10.1007/s11269-021-03016-w.

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Theory Growth

Causal Analysis

- causal compression[25], causal structure learning[26], LiNGAM-MMI[27], and time series causal network construction[28].

[25] Aleksander Wieczorek and Volker Roth. "Causal Compression". In: *arXiv preprint arXiv:1611.00261* (2016).

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Theory Growth

Machine Learning

- structure learning[29], clustering[30], nonlinear PCA[31], decision trees[32], reinforcement learning[33].

[29] Luiz Desuó Neto et al. "Learning Continuous Decomposable Models Using Mutual Information and Statistical Copulas". In: *Entropy* 28.3 (2026), p. 293. ISSN: 1099-4300. DOI: 10.3390/e28030293. URL: https://www.mdpi.com/1099-4300/28/3/293, Luiz Desuó Neto. "Untying the Gordian knot of Bayesian inference in Markov networks". PhD thesis. Universidade de São Paulo, 2025.

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Theory Growth

Neural Networks and Deep Learning

- graph neural networks[34], IEGAIN[35], neural network pruning[36], and tree-copula variational autoencoders[37].

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Theory Growth

Control Theory

- model prediction controller[38] and sliding mode controller[39].

[38] Zhiwei Li et al. "Passenger-centric model predictive control for indoor temperature to facilitate energy-efficient airport terminal". In: *Building and Environment* 283 (2025), p. 113344. ISSN: 0360-1323. DOI: https://doi.org/10.1016/j.buildenv.2025.113344. URL: https://www.sciencedirect.com/science/article/pii/S0360132325008200.

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Theory Growth

Optimization

- CE based evolutionary algorithm[40], CE based estimation of vine copula distribution algorithm[41], and CE based grey wolf optimization[42].

[40] Stuart William Card. "Towards an Information Theoretic Framework for Evolutionary Learning". PhD thesis. Syracuse University, 2011.

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Theory Growth

Approximation Theory

- B-spline approximation[43].

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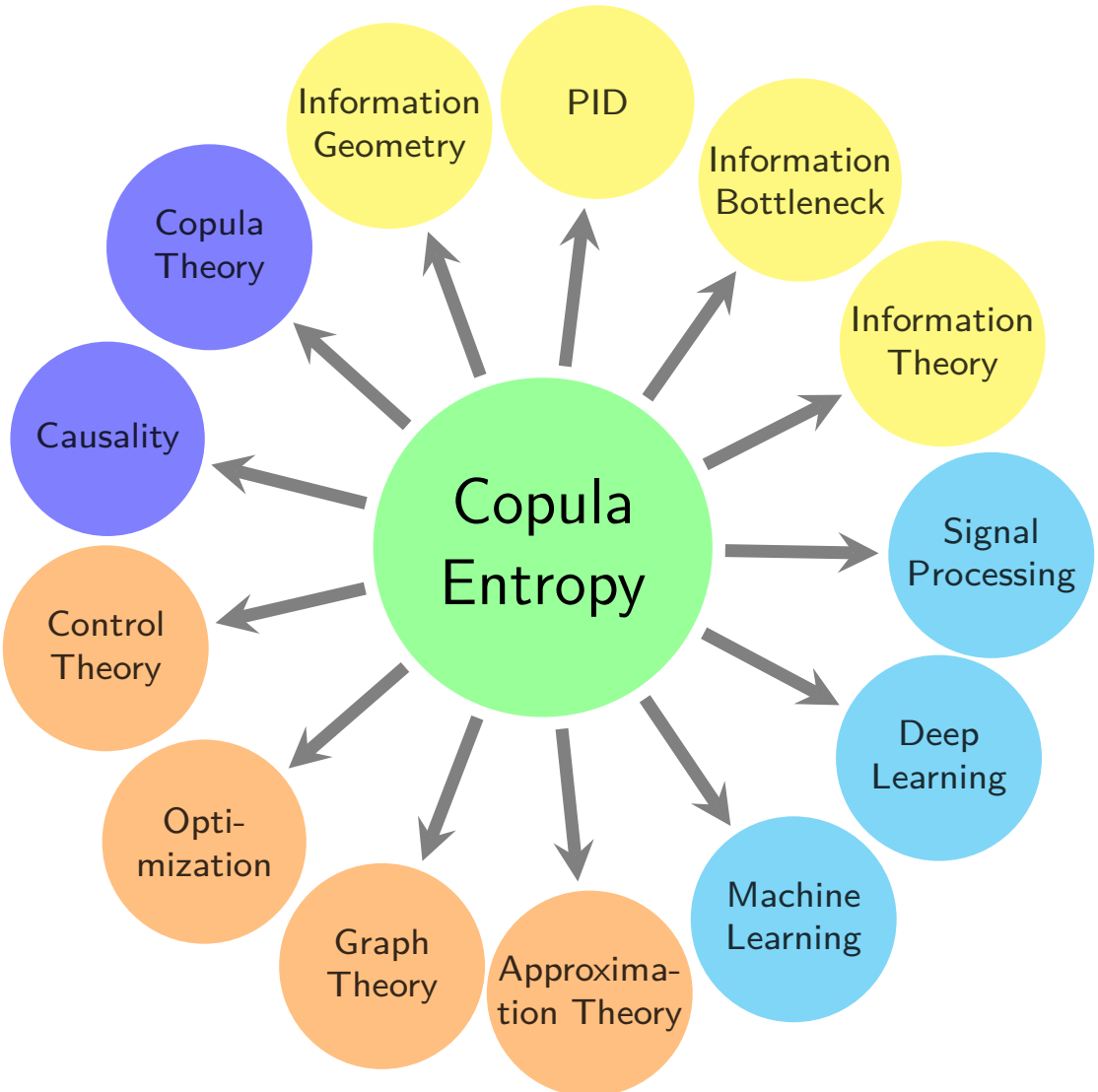
Theory Growth

Signal Processing

- copula component analysis[44].

[44] Paola Palmitesta and Corrado Provasi. "Copula component analysis for dependence modelling". In: *Quaderni di Statistica* 14 (2012), pp. 185–188.

Theory Growth



Monograph

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Thanks!

Q&A