Copula Entropy

Theory and Applications

MA Jian
majian.2003@tsinghua.org.cn

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Lecture 4: Theoretical applications (2)

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Causal Discovery[1]

[1] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

Copula Entropy: Causal Discovery

- Problem
  - To infer causality from time series data by *estimating Transfer Entropy*
- History & Significance
  - Causality is one of the oldest topics in philosophy
  - Causal discovery is a fundamental problem of all sciences
- Correlation vs Causality
  - Correlation does not mean causation
  - Correlation is symmetric while causality is asymmetric
  - Correlation is only helpful for prediction while causality implies temporal relation, intervention and control

Copula Entropy: Causal Discovery

- Wiener's Principle[2]

  Cause should improve the prediction of effect.

- Granger Causality[3]

  improvement measured by the variance of prediction error

Definition (Granger Causality)

Given time series X_t, Y_t, we say X cause Y in Granger's sense if

$$\delta^2(Y_{t+1}|Y_t, X_t) < \delta^2(Y_{t+1}|Y_t), \tag{1}$$

where δ^2 is variance of prediction error.

[2] Norbert Wiener. "The theory of prediction. Modern mathematics for engineers". In: *New York* 165.6 (1956).

[3] Clive WJ Granger. "Investigating causal relations by econometric models and cross-spectral methods". In: *Econometrica: Journal of the Econometric Society* (1969), pp. 424–438, Clive WJ Granger. "Testing for causality: a personal viewpoint". In: *Journal of Economic Dynamics and control* 2 (1980), pp. 329–352.

Copula Entropy: Causal Discovery

- Transfer Entropy[4]
  - Definition
    improvement on the uncertainty of prediction measured by Shannon entropy

$$TE = \sum p(Y_{t+1}, Y^t, X_t) \log \frac{p(Y_{t+1}|Y^t, X_t)}{p(Y_{t+1}|Y^t)} \tag{2}$$
$$= H(Y_{t+1}|Y^t) - H(Y_{t+1}|Y^t, X_t) \tag{3}$$

  - Essentially conditional mutual information

$$TE = I(Y_{t+1}; X_t | Y^t) \tag{4}$$

  - Entropy based estimation

$$TE = H(Y_{t+1}, Y_t) + H(Y_t, X_t) - H(Y_{t+1}, Y_t, X_t) - H(Y_t) \tag{5}$$

- Issue on TE
  difficult to estimate, some think impossible without model assumptions

[4] Thomas Schreiber. "Measuring information transfer". In: *Physical Review Letters* 85.2 (2000), pp. 461–464.

Copula Entropy: Causal Discovery

- TE via CE

Proposition

[a] *Transfer Entropy can be represented with only Copula Entropy.*

$$T_{x\to y} = -H_c(Y_{t+1}, Y^t, X_t) + H_c(Y_{t+1}, Y^t) + H_c(Y^t, X_t) - H_c(Y^t) \quad (6)$$

[a] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

- Non-parametric Estimator of TE
  1. estimating three or four CE terms in (6);
  2. calculating TE for these estimated CEs.
- inheriting all the merits of the non-parametric CE estimator

Copula Entropy: Causal Discovery

Experiments on the UCI Beijing PM2.5 data[5]

- Overview of the data
  - Time

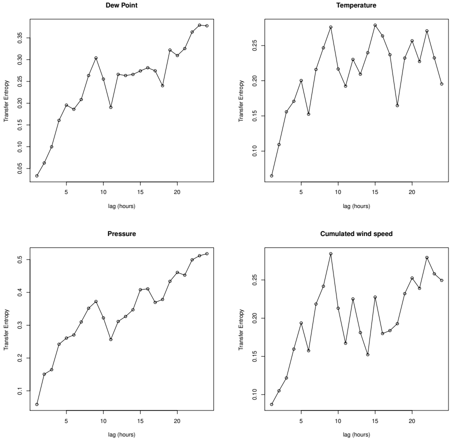
    hourly data from 2010-01-01 to 2014-12-31, which results in 43824 samples with missing values.
  - Observations
    - PM2.5 data of US Embassy in Beijing
    - Meteorological data from Beijing Capital International Airport
  - Meteorological factors

    dew point, temperature, pressure, cumulated wind speed, combined wind direction, cumulated hours of snow, cumulated hours of rain.
- Experimental data
  - the first four factors used in the experiments;
  - 1000 samples without missing values (2010-04-02∼2010-05-14).

[5] **Asuncion2007**.

Copula Entropy: Causal Discovery

Results: Effects of meteorological factors on PM2.5



- Two phrases
  - Sharp increase phrase: the first 9 hours time lag, and peak at about 9 hours lag;
  - Flat increase phrase: TE of Dew point and pressure increase with relatively flat rate while TE of temp. and cumulated wind speed does increase any more.
- Interpretation
  - The effects do not show immediately and are cumulating processes.

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Time Lag Estimation[6]

[6] Jian Ma. "Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy". In: *arXiv preprint arXiv:2301.06037*, arXiv:2301.06037 (2023). arXiv: 2301.06037 [cs.LG].

Copula Entropy: Time Lag Estimation

- Problem
  - To identify time lag in dynamical systems with copula entropy based transfer entropy
- Significance
  - Time lag is ubiquitous in physical, social, and biological systems
  - Identifying time lag is of fundamental importance in applications of dynamical systems
- Related Methods
  - Auto-correlation
  - Time-delayed mutual information

Copula Entropy: Time Lag Estimation

- System model

$$y_t = f(x_{t-l}, \mathbf{z}_t) + \epsilon \tag{7}$$

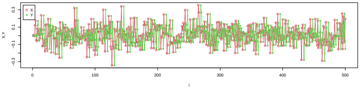
- Our method
  1. estimating transfer entropies in time lag window from data with the CE-based estimator
  2. identifying the time lag l associated with the maximal TE value

$$\hat{l} = \arg\max_{l} TE_{X \to Y}(l) \tag{8}$$

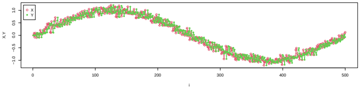
Copula Entropy: Time Lag Estimation

- Simulations
  1. generate trajectories from five simulated dynamical system with respect to different state or ouput lags
  2. identify the time lag with our method
- Simulated systems
  1. a system driven by random input with output lag
  2. a system driven by sine function with output lag
  3. Wiener process with output lag
  4. a first-order linear system with state lag
  5. a first-order nonlinear system with state lag

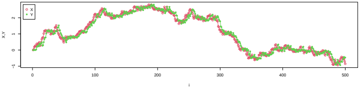
Copula Entropy: Time Lag Estimation



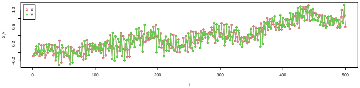
(a) random system



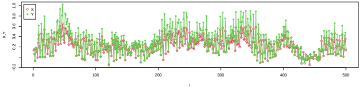
(b) sin function



(c) Wiener process



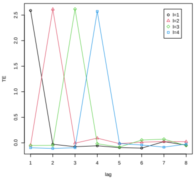
(d) linear first-order system



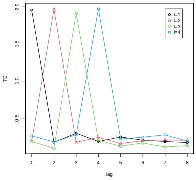
(e) nonlinear first-order system

Figure: Simulated trajectories.

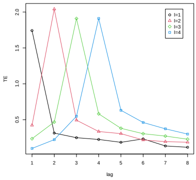
Copula Entropy: Time Lag Estimation



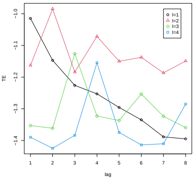
(a) random system



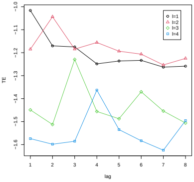
(b) sin function



(c) Wiener process



(d) linear system



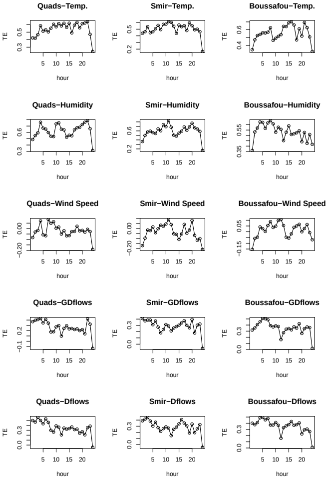
(e) nonlinear system

Figure: Simulation results of the experiments on time lag estimation.

Copula Entropy: Time Lag Estimation

Power consumption of the Tetouan city[7]

- Data
  - power consumption of 3 networks in 2017
  - weather factors, including temperature, humidity, wind speed, general diffuse flows, diffuse flows
- Power consumption forecast
  - To identify time lags from weather to power consumption



[7] **Asuncion2007**.

Copula Entropy: Application 6

System Identification[8]

[8] Jian Ma. "Two-Sample Test with Copula Entropy". In: *arXiv preprint arXiv:2307.07247* (2023). arXiv: 2307.07247 [stat.ME].

Copula Entropy: System Identification

- Problem
  - To discover differential equation from time series data
- Significance
  - differential equations are the main mathematical tools for modeling dynamical systems
  - discovering differential equations of dynamical systems has wide applications in many scientific fields
- Related Methods
  - SINDy
  - Gaussian processes

Copula Entropy: System Identification

- Idea
  considering system identification as a variable selection problem

$$\frac{dx_i}{dt} = f(\mathbf{x}, t). \tag{9}$$

- Our method
  1. calculating the derivative of system variables with differential operator;
  2. estimating the CEs between the calculated derivatives and the covariates of the system;
  3. selecting the covariates with high CE value for each derivatives.

Copula Entropy: System Identification

- Simulations
  simulating time series data from Lorenz system and Rössler system
- Results

Lorenz system

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = \rho x - y - xz$$
$$\frac{dz}{dt} = -\beta z + xy$$

Rössler system

$$\frac{dx}{dt} = -(y + z)$$
$$\frac{dy}{dt} = x + ay$$
$$\frac{dz}{dt} = b + z(x - c)$$

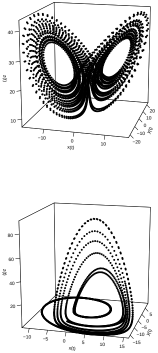


Figure: Simulated data.

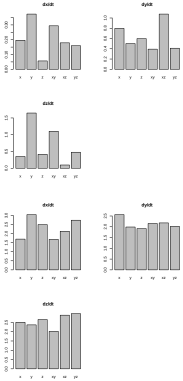


Figure: Results.

Monograph

Jian Ma. "Copula Entropy: Theory and Applications". In: *arXiv preprint arXiv:2025.18168* (2025)

Thanks!

Q&A