Copula Entropy

Theory and Applications

MA Jian
majian.2003@tsinghua.org.cn

Contents

Lecture 7: Generalizations

Tsallis Entropy

Tsallis Entropy

that generalizing Shannon entropy by extending additivity[1]

Definition (Tsallis Entropy)

Given random variables $(X, Y) \sim P(x, y)$, Tsallis entropy is defined as

$$H_T(x, y) = \frac{1}{q-1}\Big(1 - \int_x \int_y p^q(x, y)dxdy\Big), \tag{1}$$

where $q \in R$.

[1] Constantino Tsallis. "Possible generalization of Boltzmann-Gibbs statistics". In: *Journal of Statistical Physics* 52.1 (1988), pp. 479–487. ISSN: 1572-9613. DOI: 10.1007/BF01016429. URL: https://doi.org/10.1007/BF01016429, Constantino Tsallis. "The Nonadditive Entropy S_q and Its Applications in Physics and Elsewhere: Some Remarks". In: *Entropy* 13.10 (2011), pp. 1765–1804. ISSN: 1099-4300. DOI: 10.3390/e13101765. URL: https://www.mdpi.com/1099-4300/13/10/1765, Velimir M. Ilić and Miomir S. Stanković. "Generalized Shannon-Khinchin axioms and uniqueness theorem for pseudo-additive entropies". In: *Physica A: Statistical Mechanics and its Applications* 411 (2014), pp. 138–145. ISSN: 0378-4371. DOI: 10.1016/j.physa.2014.05.009. URL: http://dx.doi.org/10.1016/j.physa.2014.05.009.

Tsallis Entropy

Tsallis Entropy

If $q \to 1$, then Tsallis entropy becomes Shannon entropy:

$$\lim_{q \to 1} H_T(x, y) = H(x, y). \tag{2}$$

Tsallis Copula Entropy

Tsallis Copula Entropy[2]

Definition (Tsallis Copula Entropy)

Given random variables (X, Y) and their copula density function $c(u, v)$, Tsallis copula entropy is defined as

$$H_{TC}(x, y) = \frac{1}{q-1}\left(1 - \int_u \int_v c^q(u, v)dudv\right), \tag{3}$$

where $q \in R$.

[2] Seyedeh Azadeh Fallah Mortezanejad, Gholamreza Mohtashami Borzadaran, and Bahram sadeghpour Gildeh. "Joint dependence distribution of data set using optimizing Tsallis copula entropy". In: *Physica A: Statistical Mechanics and its Applications* 533 (2019), p. 121897. ISSN: 0378-4371. DOI: https://doi.org/10.1016/j.physa.2019.121897. URL: https://www.sciencedirect.com/science/article/pii/S0378437119311161.

Tsallis Copula Entropy

Tsallis Copula Entropy
If $q \to 1$, then Tsallis CE becomes CE:

$$\lim_{q \to 1} H_{TC}(x, y) = H_C(x, y). \tag{4}$$

Tsallis Copula Entropy

Maximal Tsallis Copula Entropy

$$\max H_{TC}(x, y) \tag{5}$$

subject to:

$$\int_u \int_v c(u, v) dudv = 1, \tag{6}$$

$$\int_u \int_v u^r c(u, v) dudv = \frac{1}{r+1}, \tag{7}$$

$$\int_u \int_v v^r c(u, v) dudv = \frac{1}{r+1}, \tag{8}$$

$$\int_u \int_v uv\ c(u, v) dudv = \frac{\rho+3}{12} \tag{9}$$

Solution:

$$c(u, v) = \sqrt[q-1]{-\frac{q-1}{q}(\lambda_0 + \sum_{r=1}^{n} (\lambda_r^u u^r + \lambda_r^v v^r) + \lambda_\rho uv)} \tag{10}$$

Rényi Entropy

Rényi Entropy

the most general entropy form that satisfies additivity[3]

Definition (Rényi Entropy)

Given random variables $\mathbf{X}$, Rényi entropy is defined as

$$H_R(\mathbf{x}) = \frac{1}{1-\alpha} \log \int_{\mathbf{x}} p^{\alpha}(\mathbf{x}) d\mathbf{x}, \tag{11}$$

where $\alpha > 0, \alpha \neq 1$.

[3] Alfréd Rényi. "On measures of entropy and information". In: *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*. Vol. 4. University of California Press. 1961, pp. 547–562.

Rényi Entropy

Rényi Entropy
If $\alpha \to 1$, then Rényi entropy degenerates to Shannon entropy:

$$\lim_{\alpha \to 1} H_R(\mathbf{x}) = H(\mathbf{x}). \tag{12}$$

Copula Rényi Entropy

Copula Rényi Entropy[4]

Definition (Copula Rényi Entropy)

Given random variables $\mathbf{X}$ and their copula density function $c(\mathbf{u})$, copula Rényi entropy is defined as

$$H_{CR}(\mathbf{x}) = \frac{1}{1-\alpha} \log \int_{\mathbf{u}} c^{\alpha}(\mathbf{u}) d\mathbf{u}, \tag{13}$$

where $\alpha > 0, \alpha \neq 1$.

[4] Shital Saha and Suchandan Kayal. "Multivariate Rényi inaccuracy measures based on copulas: properties and application". In: *arXiv preprint arXiv:2502.17215* (2025). arXiv: 2502.17215 [math.ST]. URL: https://arxiv.org/abs/2502.17215.

Copula Rényi Entropy

Copula Rényi Entropy

If $\alpha \to 1$, then copula Rényi entropy becomes CE:

$$\lim_{\alpha \to 1} H_{CR}(\mathbf{x}) = H_c(\mathbf{x}). \tag{14}$$

Survival Copula Entropy

Given random variables X, Y, **survival function** is defined as

$$\overline{F}(x,y) = P(X > x, Y > y). \tag{15}$$

Accordingly, one can define bivariate **survival copula** as

$$\overline{C}(u,v) = C(U > u, V > v). \tag{16}$$

where u, v are margins.

Survival Copula Entropy

CUmulative Residual Entropy (CURE)[5]

Definition (Cumulative Residual Entropy)

Given random variables (X, Y) and their survival function $\overline{F}(x, y)$, cumulative residual entropy is defined as

$$H_{CURE}(x, y) = -\int_x \int_y \overline{F}(x, y) \log \overline{F}(x, y) dxdy. \tag{17}$$

[5] Murali Rao et al. "Cumulative residual entropy: a new measure of information". In: *IEEE Transactions on Information Theory* 50.6 (2004), pp. 1220–1228. ISSN: 1557-9654. DOI: 10.1109/TIT.2004.828057.

Survival Copula Entropy

Survival Copula Entropy[6]

Definition (Survival Copula Entropy)

Given random variables (X, Y) and their survival copula $\overline{C}(u, v)$, survival copula entropy is defined as

$$H_{SCE}(x, y) = -\int_0^1 \int_0^1 \overline{C}(u, v) \log \overline{C}(u, v) dudv. \tag{18}$$

[6] **Sunoj2025**.

Cumulative Copula Entropy

Given random variable X, **cumulative distribution function** is defined as

$$F(x) = P(X \leq x). \tag{19}$$

Cumulative Copula Entropy

CUmulative Entropy (CUE)[7]

Definition (Cumulative Entropy)

Given random variables $\mathbf{X}$ and their cumulative distribution function $F(\mathbf{x})$, cumulative entropy is defined as

$$H_{CUE}(\mathbf{x}) = -\int_{\mathbf{x}} F(\mathbf{x}) \log F(\mathbf{x}) d\mathbf{x}. \tag{20}$$

[7] Antonio Di Crescenzo and Maria Longobardi. "On cumulative entropies". In: *Journal of Statistical Planning and Inference* 139.12 (2009), pp. 4072–4087. ISSN: 0378-3758. DOI: https://doi.org/10.1016/j.jspi.2009.05.038. URL: https://www.sciencedirect.com/science/article/pii/S0378375809001621.

Cumulative Copula Entropy

Cumulative Copula Entropy[8]

Definition (Cumulative Copula Entropy)

Given random variables $\mathbf{X}$ and their cumulative copula $C(\mathbf{u})$, cumulative copula entropy is defined as

$$H_{CCE}(\mathbf{x}) = -\int_{\mathbf{u}} C(\mathbf{u}) \log C(\mathbf{u}) d\mathbf{u}. \tag{21}$$

[8] Mohd. Arshad, Swaroop Georgy Zachariah, and Ashok Kumar Pathak. "Multivariate Information Measures: A Copula-based Approach". In: *arXiv preprint arXiv:2408.02028* (2024). arXiv: 2408.02028 [stat.ME]. URL: https://arxiv.org/abs/2408.02028.

Copula Divergence

Kullback-Leibler Divergence[9]

Definition (Kullback-Leibler Divergence)

Given probability density functions p_X, p_Y, KL divergence is defined as

$$D_{KL}(p_X, p_Y) = \int_x p_X(x) \log \frac{p_X(x)}{p_Y(x)} dx. \tag{22}$$

[9] Solomon Kullback and Richard A Leibler. "On information and sufficiency". In: *The Annals of Mathematical Statistics* 22.1 (1951), pp. 79–86.

Copula Divergence

Rényi Divergence[10]

Definition (Rényi Divergence)

Given probability density functions p_X, p_Y, Rényi divergence is defined as

$$D_R(p_X, p_Y) = \frac{1}{\alpha - 1} \log \int_x p_X^{\alpha}(x) p_Y^{1-\alpha}(x) dx, \tag{23}$$

where $\alpha > 0, \alpha \neq 1$.

[10] Alfréd Rényi. "On measures of entropy and information". In: *Proceedings of the fourth Berkeley symposium on mathematical statistics and probability, volume 1: contributions to the theory of statistics*. Vol. 4. University of California Press. 1961, pp. 547–562.

Copula Divergence

Tsallis Divergence[11]

Definition (Tsallis Divergence)

Given probability density functions p_X, p_Y, Tsallis divergence is defined as

$$D_T(p_X, p_Y) = \frac{1}{q-1}\Big(\int_x p_X^q(x) p_Y^{1-q}(x)dx - 1\Big), \tag{24}$$

where $q > 0, q \neq 1$.

[11] Lisa Borland, Angel R. Plastino, and Constantino Tsallis. "Information gain within nonextensive thermostatistics". In: *J. Math. Phys.* 39.12 (1998), pp. 6490–6501. ISSN: 0022-2488. URL: https://doi.org/10.1063/1.532660, S. Furuichi, K. Yanagi, and K. Kuriyama. "Fundamental properties of Tsallis relative entropy". In: *J. Math. Phys.* 45.12 (2004), pp. 4868–4877. ISSN: 0022-2488. URL: https://doi.org/10.1063/1.1805729.

Copula Divergence

Jeffreys Divergence[12]

Definition (Jeffreys Divergence)

Given probability density functions p_X, p_Y, Jeffreys divergence is defined as

$$D_J(p_X, p_Y) = \int_x (p_X(x) - p_Y(x))(\log p_X(x) - \log p_Y(x))dx. \tag{25}$$

[12] Harold Jeffreys. "An invariant form for the prior probability in estimation problems". In: *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 186.1007 (1946), pp. 453–461. DOI: 10.1098/rspa.1946.0056. eprint: https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.1946.0056. URL: https://royalsocietypublishing.org/doi/abs/10.1098/rspa.1946.0056.

Copula Divergence

Hellinger Divergence[13]

Definition (Hellinger Divergence)

Given probability density functions p_X, p_Y, Hellinger divergence is defined as

$$D_H(p_X, p_Y) = \int_x (\sqrt{p_X(x)} - \sqrt{p_Y(x)})^2 dx. \tag{26}$$

[13] Ernst Hellinger. "Neue Begründung der Theorie quadratischer Formen von unendlichvielen Veränderlichen". In: *Journal für die reine und angewandte Mathematik* 136 (1909), pp. 210–271. DOI: doi:10.1515/crll.1909.136.210. URL: https://doi.org/10.1515/crll.1909.136.210.

Copula Divergence

Copula Rényi Divergence[14]

Definition (Copula Rényi Divergence)

Given random variables (X, Y) and their copula density function $c(u, v)$, copula Rényi divergence is defined as

$$D_{CR}(c_{XY}) = \frac{1}{\alpha - 1} \log \int_u \int_v c^{\alpha}(u, v) du dv, \tag{27}$$

where $\alpha > 0, \alpha \neq 1$.

[14] Morteza Mohammadi and Mahdi Emadi. "Nonparametric tests of independence using copula-based Renyi and Tsallis divergence measures". In: *Statistics, Optimization & Information Computing* 11.4 (2023), pp. 949–962. DOI: 10.19139/soic-2310-5070-1691. URL: http://www.iapress.org/index.php/soic/article/view/1691.

Copula Divergence

Copula Tsallis Divergence[15]

Definition (Copula Tsallis Divergence)

Given random variables (X, Y) and their copula density function $c(u, v)$, copula Tsallis divergence is defined as

$$D_{CT}(c_{XY}) = \frac{1}{q-1}\Big(\int_u \int_v c^q(u, v) du dv - 1\Big), \tag{28}$$

where $q > 0, q \neq 1$.

[15] Morteza Mohammadi and Mahdi Emadi. "Nonparametric tests of independence using copula-based Renyi and Tsallis divergence measures". In: *Statistics, Optimization & Information Computing* 11.4 (2023), pp. 949–962. DOI: 10.19139/soic-2310-5070-1691. URL: http://www.iapress.org/index.php/soic/article/view/1691.

Copula Divergence

Copula Jeffreys Divergence[16]

Definition (Copula Jeffreys Divergence)

Given random variables (X, Y) and their copula density function $c(u, v)$, copula Jeffreys divergence is defined as

$$D_{CJ}(c_{XY}) = \int_u \int_v (c(u, v) - 1) \log c(u, v) du dv. \tag{29}$$

[16] Morteza Mohammadi, Mahdi Emadi, and Mohammad Amini. "Bivariate Dependency Analysis using Jeffrey and Hellinger Divergence Measures based on Copula Density Estimation by Improved Probit Transformation". In: *Journal of Statistical Sciences* 15.1 (2021), pp. 233–254. DOI: 10.29252/jss.15.1.233.

Copula Divergence

Copula Hellinger Divergence[17]

Definition (Copula Hellinger Divergence)

Given random variables (X, Y) and their copula density function $c(u, v)$, copula Hellinger divergence is defined as

$$D_{CH}(c_{XY}) = \int_u \int_v (\sqrt{c(u, v)} - 1)^2 dudv. \tag{30}$$

[17] Morteza Mohammadi, Mahdi Emadi, and Mohammad Amini. "Bivariate Dependency Analysis using Jeffrey and Hellinger Divergence Measures based on Copula Density Estimation by Improved Probit Transformation". In: *Journal of Statistical Sciences* 15.1 (2021), pp. 233–254. DOI: 10.29252/jss.15.1.233.

Copula Inaccuracy

Inaccuracy[18]
measuring the distance between distributions

Definition (Inaccuracy)

Given random variables $\mathbf{X}, \mathbf{Y} \in R^n$ and their density functions $p_{\mathbf{X}}, p_{\mathbf{Y}}$, inaccuracy is defined as

$$D_I(p_{\mathbf{X}}, p_{\mathbf{Y}}) = -\int_{\mathbf{x}} p_{\mathbf{X}}(\mathbf{x}) \log p_{\mathbf{Y}}(\mathbf{x}) d\mathbf{x}. \tag{31}$$

[18] David F Kerridge. "Inaccuracy and inference". In: *Journal of the Royal Statistical Society. Series B (Methodological)* 23.1 (1961), pp. 184–194.

Copula Inaccuracy

Inaccuracy[19]

- relation to entropy

$$D_I(p_{\mathbf{X}}, p_{\mathbf{Y}}) \geq H(\mathbf{x}) \tag{32}$$

- relation to divergence

$$D_{KL}(p_{\mathbf{X}}, p_{\mathbf{Y}}) = D_I(p_{\mathbf{X}}, p_{\mathbf{Y}}) - H(\mathbf{x}). \tag{33}$$

[19] David F Kerridge. "Inaccuracy and inference". In: *Journal of the Royal Statistical Society. Series B (Methodological)* 23.1 (1961), pp. 184–194.

Copula Inaccuracy

Copula Inaccuracy[20]

Definition (Copula Inaccuracy)

Given random variables $\mathbf{X}, \mathbf{Y} \in R^n$ and their copula density functions $c_{\mathbf{X}}(\mathbf{u}), c_{\mathbf{Y}}(\mathbf{u})$, copula inaccuracy is defined as

$$D_{CI}(c_{\mathbf{X}}, c_{\mathbf{Y}}) = \int_{\mathbf{u}} c_{\mathbf{X}}(\mathbf{u}) \log c_{\mathbf{Y}}(\mathbf{u}) d\mathbf{u}. \tag{34}$$

[20] Aman Pandey and Chanchal Kundu. "Copula-Based Modeling of Fractional Inaccuracy: A Unified Framework". In: *arXiv preprint arXiv:2506.19748* (2025). arXiv: 2506.19748 [math.ST]. URL: https://arxiv.org/abs/2506.19748.

Copula Extropy

Extropy[21]
the dual to entropy

Definition (Extropy)

Given random variables $(X, Y) \sim P(x, y)$, extropy is defined as

$$J(x, y) = \frac{1}{4} \int_0^{\infty} \int_0^{\infty} p^2(x, y) dxdy. \tag{35}$$

[21] Frank Lad, Giuseppe Sanfilippo, and Gianna Agrò. "Extropy: Complementary Dual of Entropy". In: *Statistical Science* 30.1 (2015), pp. 40–58. DOI: 10.1214/14-STS430. URL: https://doi.org/10.1214/14-STS430.

Copula Extropy

Copula Extropy[22]

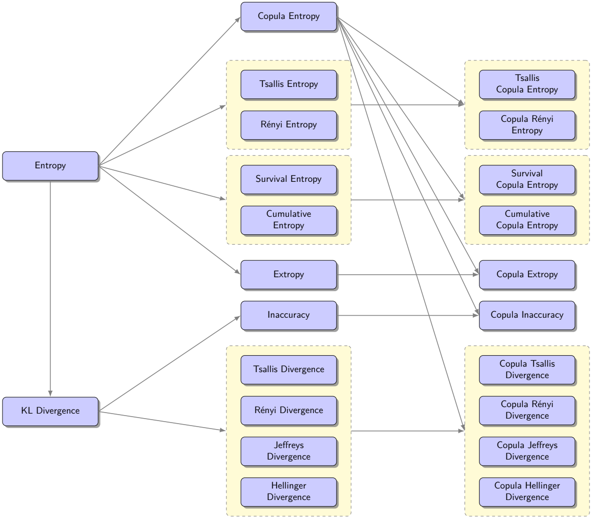
Definition (Copula Extropy)

Given random variables (X, Y) and their copula density function $c(u, v)$, copula extropy is defined as

$$J_c(x, y) = \frac{1}{4} \int_0^1 \int_0^1 c^2(u, v) du dv. \tag{36}$$

[22] Shital Saha and Suchandan Kayal. "Copula-based extropy measures, properties and dependence in bivariate distributions". In: *Communications in Statistics - Theory and Methods* 55.1 (2026). See also arXiv preprint arXiv:2311.08061., pp. 189–216. arXiv: 2311.08061 [math.ST].

Relationship between entropies and divergences



Monograph

Jian Ma. "Copula Entropy: Theory and Applications". In: *arXiv preprint arXiv:2025.18168* (2025)

Thanks!

Q&A