# Copula Entropy

## Theory and Applications

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# Copula Theory

## Definition (Copula)

[a] Given $N$ random variables $\mathrm{X} = (X_1, \ldots, X_N) \in \mathcal{R}^N$. Let $\{u_i = F_i(x_i), i = 1, \ldots, N\}$ be the marginal distribution functions of X. A $N$-dimensional copula $C : \mathcal{I}^N \to \mathcal{I}(\mathcal{I} = [0, 1])$ of X is a function with following properties:

1. $C$ is grounded and $N$-increasing;
2. $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$.

[a] Roger B Nelsen. *An introduction to copulas*. Springer, 2007.

- the theory on **representation** of statistical dependence in probability
- copula function contains all the dependence information between random variables
- a probability function on unit cubic

# Copula Theory

**Theorem (Sklar's Theorem)**

[a] *Given a random vector* $\mathrm{X} = (X_1, \ldots, X_N)$, *its CDF* $\mathrm{F}(\mathrm{x})$ *can be represented as*

$$\mathrm{F}(\mathrm{x}) = C(u_1, \ldots, u_N), \tag{1}$$

*where* $C$ *is a copula function,* $\{u_i\}$ *are marginal distribution functions of* $\mathrm{X}$. *If* $\{F_i\}$ *are continuous, then* $C$ *is unique.*

[a] M. Sklar. "Fonctions de repartition an dimensions et leurs marges". In: *Publ. Inst. Statist. Univ. Paris* 8 (1959), pp. 229–231.

- the core of copula theory
- there exists a copula function for each multivariate probability function

# Copula Theory

**Corollary**

*The probabilistic density function (PDF) $p(\mathrm{x})$ of X can be represented as*

$$p(\mathrm{x}) = c(\mathrm{u}) \prod_{i=1}^{N} p_i(x_i), \quad (2)$$

*where $\{p_i, i = 1, \ldots, N\}$ are marginal density functions of X, and $c$ is copula density.*

- seperating dependence representation with properties of individual variables

# Copula Entropy: Theory

**Definition (Copula Entropy)**

Let X be random variables with marginals u and copula density $c(\mathbf{u})$. Copula Entropy of X is defined as

$$H_c(\mathbf{x}) = -\int_{\mathbf{u}} c(\mathbf{u}) \log c(\mathbf{u}) d\mathbf{u}. \tag{3}$$

- a special type of Shannon entropy
- an ideal measure of statistical independence
- distribution-free

# Copula Entropy: Theory

**Theorem**

*Mutual Information of* X *is equavalent to its negative copula entropy.*

$$I(\mathrm{x}) = -H_c(\mathrm{x}). \tag{4}$$

**Corollary**

$$H(\mathrm{x}) = \sum_i H_i(x_i) + H_c(\mathrm{x}). \tag{5}$$

- the bridge between copula theory and information theory[1]

[1] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

# Copula Entropy: Theory

- Axiomatic Properties of Copula Entropy
  - multivariate
  - symmetric
  - non-negative, 0 iff independence
  - invariant to monotonic transformation
  - equivalent to correlation coefficient in Gaussian cases
- An ideal measure compared with others

Table: Comparison with other independence measures.

| | Copula Entropy | Distance Correlation | HSIC |
|---|---|---|---|
| Definition | copula based | generalised corr | corr in RKHS |
| Multivariate | Yes | distance multivariance | dHSIC |
| Invariance | monotonic trans | No | No |
| Gaussanity | equivalent to cc | unclear | unclear |
| Computation | low | high | high |

# Copula Entropy: Estimation

- **Non-Parametric** Estimation Method[2]
  1. estimating empirical copula density with rank statistics
  2. estimating copula entropy with kNN entropy estimation method
- Advantages
  - distribution-free, non-parametric
  - tuning-free, insensitive to parameters
  - good convergence
  - easy to implement
  - low computation burden

[2] Jian Ma and Zengqi Sun. "Mutual information is copula entropy". In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54.

# Copula Entropy: Application I

Association Discovery[3]

[3] Jian Ma. "Discovering Association with Copula Entropy". In: *arXiv preprint arXiv:1907.12268* (2019).

# Copula Entropy: Association Discovery

- Problem
  - To discover association relationship between random variables from data
- History
  - An old and fundamental problem since statistics birth
- Related Methods
  - Pearson Correlation Coefficient
  - Regression

# Copula Entropy: Association Discovery

- Traditional association measures
  - Pearson Correlation Coefficient

$$r_{XY} = corr(X, Y) = \frac{cov(X, Y)}{\delta_X \delta_Y} \quad (6)$$

  - Spearman's $\rho$ and Kendall's $\tau$

$$\rho_{XY} = 12 \int_u \int_v C(u, v) du dv - 3 \quad (7)$$

$$\tau_{XY} = 4 \int_u \int_v C(u, v) dC(u, v) - 1 \quad (8)$$

- Why Copula Entropy?

Table: Theoretical comparison between CE and CC.

| | CC | CE |
|---|---|---|
| linearity | linear | nonlinear |
| Order | 2 | $\geq 2$ |
| Assumption | Gaussian | None |
| variate | bivariate | multivariate |

# Copula Entropy: Association Discovery

Experiments on the NHANES data

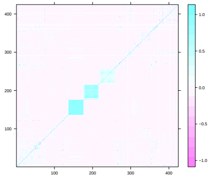
- Objectives of NHANES
  - to monitor trends and emerging issues of population health
  - to investigate its relationship with risk factors, nutritions and environmental exposures, etc.
- NHANES (2013-2014)
  - 14,332 persons from 30 different survey locations were selected;
  - Of those selected, 10,175 interviewed and 9,813 examined;
  - 5 groups of data: demographics, dietary, examination, laboratory, and questionnaire.
- Experimental data

  The laboratory data, which includes 423 variables from blood, urine, oral rinse and vaginal/Penile swabs.
- Missing values

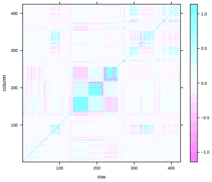
  The missing values were filled with the mean of their corresponding variables.

# Copula Entropy: Association Discovery

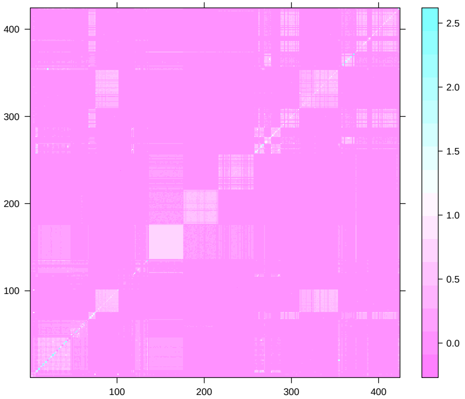
- Results - Correlation matrices



Pearson's $r$




Spearman's $\rho$




Copula Entropy

# Copula Entropy: Association Discovery

- Results - Variable groups with meanings

Table: Variable groups with biomedical meanings discovered with CE.

| Group | Index | Variables |
|---|---|---|
| 1 | 288-302 | Polycyclic Aromatic Hydrocarbones (PAH) - Urine |
| | 68-75 | Copper, Selenium & Zinc - Serum |
| | 395-420 | Urine Metals |
| 2 | 358-373 | Blood Lead, cadmium, total Mercury, Selenium, and Manganese |
| | 269-276 | Blood mercury: inorganic, ethyl and methyl |
| 3 | 277-287 | Oral Glucose Tolerance Test |
| | 258-262 | Insulin |
| | 7-9 | Cholesterol-LDL, Triglyceride&Apoliprotein(ApoB), WTSAF2YR-Fasting Subsample 2 Year MEC Weight, LBXAPB-Apolipoprotein (B) (mg/dL), LBDAPBSI-Apolipoprotein (B) (g/L) |
| 4 | 10-46 | Standard Biochemistry Profile |
| | 137-176 | Human Papillomavirus (HPV) - Oral Rinse |
| 5 | 76-101 | Personal Care and Consumer Product chemicals and Metabolites |
| | 327-353 | Phthalates and Plasticizers Metabolites - Urines |

# Copula Entropy: Application II

Structure Learning[4]

[4] Jian Ma and Zengqi Sun. "Dependence structure estimation via copula". In: *arXiv preprint arXiv:0804.4451* (2008).

# Copula Entropy: Structure Learning

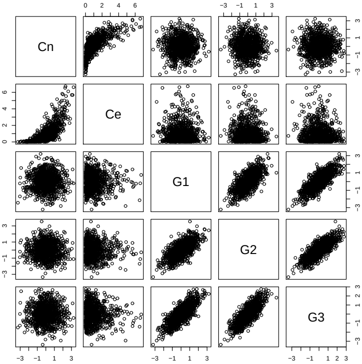
- Problem
  - To learn statistical structure among random variables from data
- Graph Representation
  - A probability density is represented with a directed or indirected graph, of which each node represents a random variable, and each edge represents a (conditional) dependence relation between two random variables
- Related Methods
  - Chow-Liu Algorithm

# Copula Entropy: Structure Learning

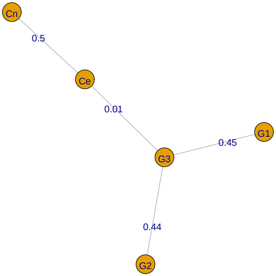
- Our Algorithm
  1. computing dependence matrix $W_x$ of data $x$ with CE estimation
  2. constructing dependence structure $T$ from $W_x$ with MST algorithm
- Advantages
  - distribution-free, non-parametric
  - tuning-free, insensitive to parameters
  - easy to implement
  - low computation burden

# Copula Entropy: Structure Learning

- Simulated Experiment
  5 random variables: the first three are Gaussian and the others two are governed by Gaussian copula with margins as normal distribution and exponential distribution respectively




Simulated data

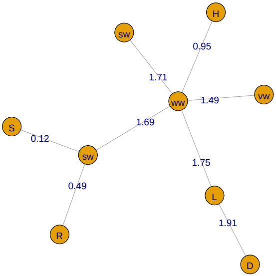


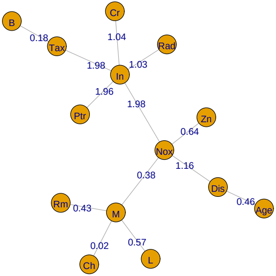

Learned graph

# Copula Entropy: Structure Learning

Experiment on real data

- Abalone data
  Predicting the age of abalone from physical measurements
- Boston housing data
  Concerns housing values in suburbs of Boston



# Copula Entropy: Application III

Variable Selection[5]

[5] Jian Ma. "Variable Selection with Copula Entropy". In: *Chinese Journal of Applied Probability and Statistics* 37.4 (2021), pp. 405–420.

# Copula Entropy: Variable Selection

- Problem
  - To select a 'right' subset of variables from the whole group for building classification or regression models with good predictability and interpretability
- History
  - An old and basic problem in statistics and machine learning
- Related Problems
  - Feature Selection
  - Model Selection

# Copula Entropy: Variable Selection

Existing methods - Likelihood with penalty

- Information Criteria
  with penalty on the number of parameters in the models

$$\mathrm{AIC} = -2L + 2p \tag{9}$$

$$\mathrm{BIC} = -2L + p \log N \tag{10}$$

- Penalized GLMs
  with penalty on the nonzero coefficients in the GLMs
  - LASSO
  - Ridge Regression
  - Elastic Net

$$\min_{\beta}\{L(\beta; y, \mathrm{X}) + \lambda_1 ||\beta||_1 + \lambda_2 ||\beta||_2^2\} \tag{11}$$

  - Adaptive LASSO

$$\min_{\beta}\{L(\beta; y, \mathrm{X}) + \lambda \sum_{j=1}^{p} w_j |\beta_j|\} \tag{12}$$

# Copula Entropy: Variable Selection

Existing methods - Statistical independence measures

- Distance Correlation

$$\mathrm{dCor}(X,Y) = \frac{\nu^2(X,Y)}{\sqrt{\nu^2(X)\nu^2(Y)}}, \qquad (13)$$

  where $\nu^2(X,Y)$ be distance covariance.

- Hilbert-Schmidt Independence Criterion (HSIC)

$$\mathrm{dHSIC}(P(\mathrm{X})) = ||\Pi(P(X_1)\otimes,\ldots,\otimes P(X_d)) - \Pi(P(\mathrm{X}))||, \qquad (14)$$

  where $\Pi$ be the mean embedding function associated with kernel functions.

# Copula Entropy: Variable Selection

- CE based method
  To select variables based on ranks of their negative CE values with target
- Advantages
  - model-free, non-parametric
  - tunning-free, insensitive to parameters
  - interpretable with physical meanings
  - supported by rigorous math
  - science instead of art, compared with existing methods
  - easy to implement, low computation burden

# Copula Entropy: Variable Selection

Experiments on the UCI heart disease data[6]

- Overview of the data
  The data set contains 4 databases (899 samples) concerning heart disease diagnosis. All attributes are numeric-valued. The data was collected from the four following locations:
  - Cleveland clinic foundation;
  - Hungarian Institute of Cardiology, Budapest;
  - V.A. medical center, long beach, CA;
  - University hospital, Zurich, Switzerland.
- Attributes
  The data has 76 attributes (#58 'num' for diagnosis). Of them, 13 attributes are recommended by professionals as clinical relevant.

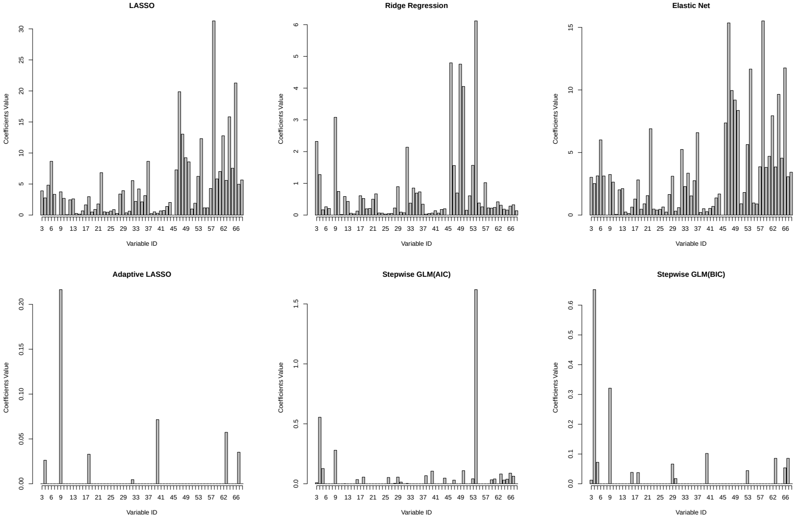
Table: Recommended attributes.

| **ID** | 3 | 4 | 9 | 10 | 12 | 16 | 19 |
|---|---|---|---|---|---|---|---|
| **Name** | age | sex | cp | trestbps | chol | fbs | restecg |
| **ID** | 32 | 38 | 40 | 41 | 44 | 51 | 58 |
| **Name** | thalach | exang | oldpeak | slope | ca | thal | num |

[6] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

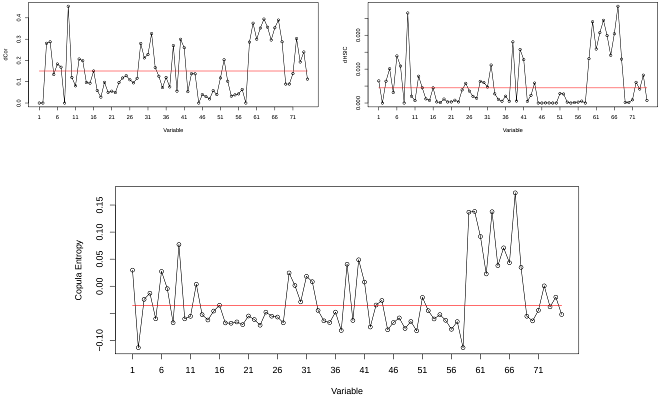
# Copula Entropy: Variable Selection

- Results - Coefficients of penalized likelihood based models



# Copula Entropy: Variable Selection

- Results - with statistical dependence measures (dCor, dHSIC, CE)



# Copula Entropy: Variable Selection

- Results - Prediction accuracy
  the selected variables present the best prediction accuracy.

| Model | Accuracy(%) |
|---|---|
| SVM(Recommended variables) | 84.20 |
| SVM(CE) | **84.76** |
| SVM(dCor) | 82.76 |
| SVM(dHSIC) | 84.54 |
| Stepwise GLM(AIC) | 51.8 |
| Stepwise GLM(BIC) | 49.1 |
| LASSO | 79.2 |
| Ridge Regression | 63.0 |
| Elastic Net | 75.9 |
| Adaptive LASSO | 35.7 |

# Copula Entropy: Variable Selection

- Results - Selected variables

  Copula Entropy selects more 'right' variables than the other methods do.

| Method | Selected Variables' ID | ✓ |
|---|---|---|
| Recommended variables | 3,4,9,10,12,16,19,32,38,40,41,44,51 | 13 |
| CE | 3,4,6,7,9,12,16,28-32,38,40,41,44,51,59-68 | **11** |
| dHSIC | 3,4,6,7,9,12,13,16,25,29-32,38,40,41,44,59-68 | 10 |
| dCor | 3,4,6,7,9,12,13,16,28-33,38,40,41,52,59-68 | 9 |
| Stepwise GLM(AIC) | 3,4,5,9,12,16,18,20,26,29,30,32,40,44,47,50,53,54,60,61,63,65-67 | 8 |
| Stepwise GLM(BIC) | 3,4,5,9,16,18,29,30,40,53,63,66,67 | 5 |
| Adaptive LASSO | 4,6,9,18,32,40,63,67 | 4 |
| LASSO<br>Ridge Regression<br>Elastic Net | all except 8,45 | - |

# Copula Entropy: Application IV

Causal Discovery[7]

[7] Jian Ma. "Estimating Transfer Entropy via Copula Entropy". In: *arXiv preprint arXiv:1910.04375* (2019).

# Copula Entropy: Causal Discovery

- Problem
  - To infer causality from time series data by *estimating Transfer Entropy*
- History & Significance
  - Causality is one of the oldest topics in philosophy.
  - Causal discovery is a central problem of all sciences.
- Correlation vs Causality
  - Correlation does not mean causation.
  - Correlation is only helpful for prediction while causality means intervention and control.

# Copula Entropy: Causal Discovery

- Causality measures
  - Wiener's Principle
    Cause should improve the prediction of effect.
  - Granger Causality
    improvement measured by the variance of prediction error

$$\delta^2(Y_{t+1}|Y_t, X_t) < \delta^2(Y_{t+1}|Y_t) \tag{15}$$

  - Transfer Entropy
    improvement on the uncertainty of prediction measured by Shannon entropy

$$TE = \sum p(Y_{t+1}, Y^t, X_t) \log \frac{p(Y_{t+1}|Y^t, X_t)}{p(Y_{t+1}|Y^t)} \tag{16}$$

$$= H(Y_{t+1}|Y^t) - H(Y_{t+1}|Y^t, X_t) \tag{17}$$

$$= I(Y_{t+1}, X_t|Y^t) \tag{18}$$

- Issue on TE
  difficult to estimate, some think impossible without model assumptions

# Copula Entropy: Causal Discovery

- TE via CE

**Proposition**

*Transfer Entropy can be represented with only Copula Entropy.*

$$T_{x\rightarrow y} = -H_c(Y_{t+1}, Y^t, X_t) + H_c(Y_{t+1}, Y^t) + H_c(Y^t, X_t) - H_c(Y^t) \quad (19)$$

- Non-parametric Estimator of TE
  1. estimating three or four CE terms in (19);
  2. calculating TE for these estimated CEs.
- inheriting all the merits of non-parametric CE estimation

# Copula Entropy: Causal Discovery

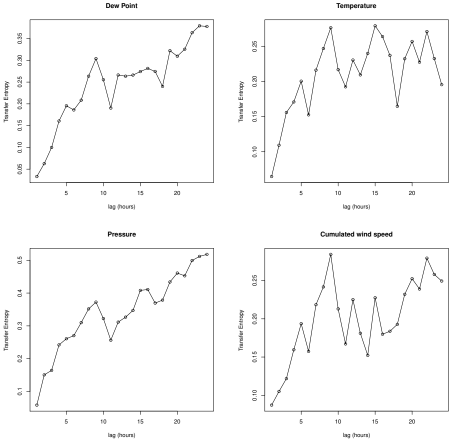
Experiments on the UCI Beijing PM2.5 data[8]

- Overview of the data
  - Time
    hourly data from 2010-01-01 to 2014-12-31, which results in 43824 samples with missing values.
  - Observations
    - PM2.5 data of US Embassy in Beijing
    - Meteorological data from Beijing Capital International Airport
  - Meteorological factors
    dew point, temperature, pressure, cumulated wind speed, combined wind direction, cumulated hours of snow, cumulated hours of rain.
- Experimental data
  - the first four factors used in the experiments;
  - 1000 samples without missing values (2010-04-02∼2010-05-14).

[8] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

# Copula Entropy: Causal Discovery

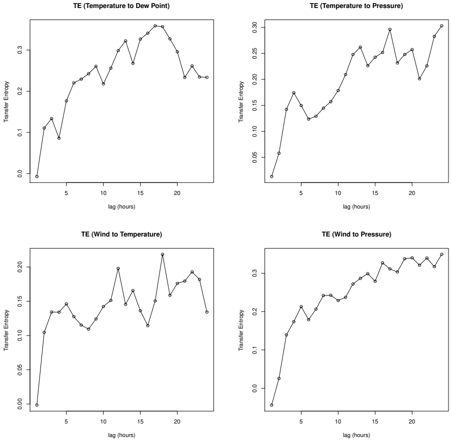
Results: Effects of meteorological factors on PM2.5




- Two phrases
  - Sharp increase phrase: the first 9 hours time lag, and peak at about 9 hours lag;
  - Flat increase phrase: TE of Dew point and pressure increase with relatively flat rate while TE of temp. and cumulated wind speed does increase any more.
- Interpretation
  - The effects do not show immediately and are cumulating processes.

# Copula Entropy: Causal Discovery

Results - Effects between meteorological factors




- Temp. to Dew Point & Pressure
- Wind to Temp. & Pressure
  - Wind changes temperature in 3 hours later and
  - Wind changes pressure in 5 hours later.

# Copula Entropy: Application V

Time Lag Estimation[9]

[9] Jian Ma. "Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy". In: *arXiv preprint arXiv:2301.06037* (2023).

# Copula Entropy: Time Lag Estimation

- Problem
  - To identify time lag in dynamical systems with copula entropy based transfer entropy
- Significance
  - Time lag is ubiquitous in physical, social, and biological systems.
  - Identifying time lag is of fundamental importance in applications of dynamical systems.
- Related Methods
  - Auto-correlation
  - Time-delayed mutual information

# Copula Entropy: Time Lag Estimation

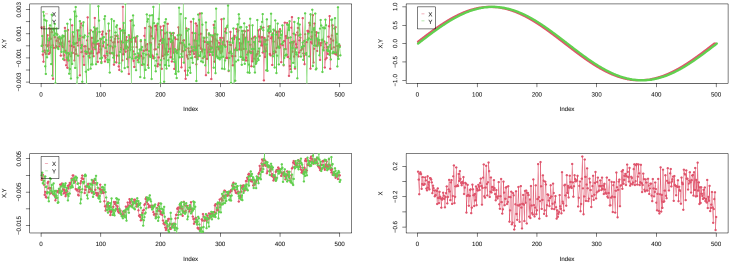
- Our method
  1. estimating transfer entropies on time lag horizon from data with the CE-based estimator
  2. identifying the time lag associated with the maximum TE value

# Copula Entropy: Time Lag Estimation

- Simulations
  1. generate trajectories from four simulated dynamical system with respect to different state or ouput lags
  2. identify the time lag with our method
- Simulated systems
  - a system driven by random walk with output lag
  - a system driven by sine function with output lag
  - Wiener process with output lag
  - a first-order linear system with state lag

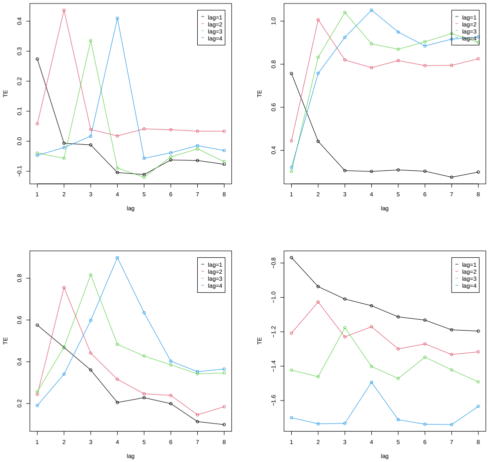
# Copula Entropy: Time Lag Estimation

- Simulated trajectories



# Copula Entropy: Time Lag Estimation

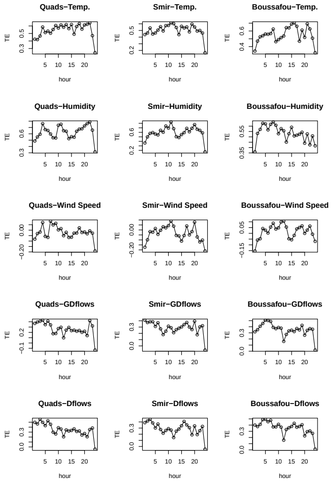
- Simulation: Results



# Copula Entropy: Time Lag Estimation

Power consumption of the Tetouan city[10]

- Data
  - power consumption of 3 networks in 2017
  - weather factors, including temperature, humidity, wind speed, general diffuse flows, diffuse flows
- Power consumption forecast
  - To identify time lags from weather to power consumption




[10] Arthur Asuncion and David Newman. *UCI machine learning repository*. 2007.

# Copula Entropy: Application VI

System Identification[11]

[11] Jian Ma. "System Identification with Copula Entropy". In: *arXiv preprint arXiv:2304.12922* (2023).

# Copula Entropy: System Identification

- Problem
  - To discover differential equation from time series data
- Significance
  - differential equations are the main mathematical tools for modelling dynamical systems.
  - discovering differential equations of dynamical systems has wide applications in many scientific fields.
- Related Methods
  - SINDy
  - Gaussian processes

# Copula Entropy: System Identification

- Idea
  considering system identification as a variable selection problem

$$\frac{dx_i}{dt} = f(\mathrm{x}, t). \tag{20}$$

- Our method
  1. calculating the derivative of system variables with differential operator;
  2. estimating the CEs between the calculated derivatives and the covariates of the system;
  3. selecting the covariates with high CE value for each derivatives.

# Copula Entropy: System Identification

- Simulations
  1. simulating time series data from the 3D Lorenz system
  2. identifying the system equation from data with our method
- Results

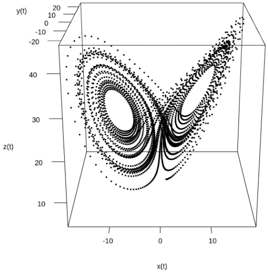



Figure: 3D plot of the simulated data.

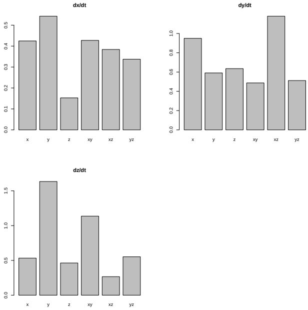



Figure: Identification results.

# Copula Entropy: Application VII

Multivariate Normality Test[12]

[12] Jian Ma. "Multivariate Normality Test with Copula Entropy". In: *arXiv preprint arXiv:2206.05956* (2022).

# Copula Entropy: Multivariate Normality Test

- Problem
  - To test the hypothesis that the distribution of data is normal distribution
- Significance
  - Normal distribution is the most important distribution in probability theory;
  - Normality is a common assumption of many statistical tools;
  - Testing normality is widely needed in real applications.
- Related Methods
  - characteristics function based
  - moments based
  - skewness and kurtosis
  - energy distance based
  - entropy based
  - Wasserstein distance based

# Copula Entropy: Multivariate Normality Test

- The proposed statistic

$$T_{ce} = H_c(\mathrm{x}) - H_c(\mathrm{x}_n), \tag{21}$$

where $\mathrm{x}_n$ is the Gaussian random vector with the same covariances as $\mathrm{x}$.

  - defined as the difference of copula entropies
  - $T_{ce} = 0$ if normal distributions

- The estimator
  - the first term in (21) can be estimated with the non-parametric CE estimator;
  - the second term in (21) can be estimated easily by first estimating the covariances $V_x$ of $\mathrm{x}$ and then calculating the result according to (22).

$$H_c(\mathrm{x}_n) = \frac{1}{2}\log|V_x|. \tag{22}$$

# Copula Entropy: Multivariate Normality Test

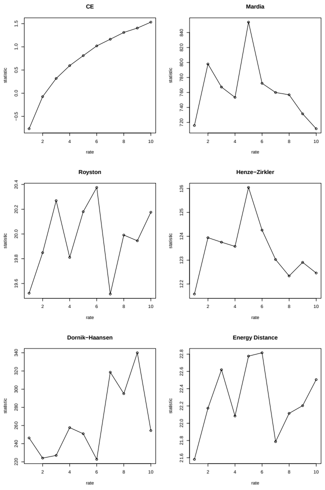
Simulation Experiments

- Data
  - bivariate normal copula with normal and exponential marginals
  - bivariate Gumbel copula with normal marginals
- Compared methods
  - Mardia's
  - Royston's
  - Henze and Zirkler's
  - Doornik and Hansen's, and
  - the energy distance based test by Rizzo and Székely

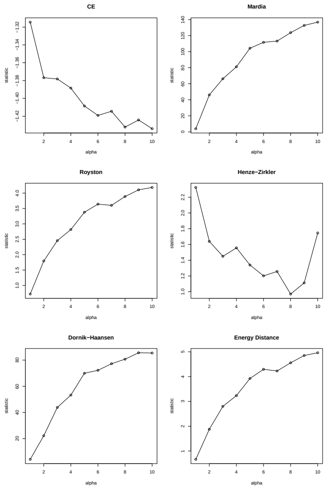
# Copula Entropy: Multivariate Normality Test

## Simulation Results

- Bivariate normal copula




- Bivariate Gumbel copula



# Copula Entropy: Application VIII

## Two-Sample Test[13]

[13] Jian Ma. "Two-Sample Test with Copula Entropy". In: *arXiv preprint arXiv:2307.07247* (2023).

# Copula Entropy: Two-Sample Test

- Problem
  - To test the hypothesis that two samples are from a same distribution
- Significance
  - a basic hypothesis testing problem;
  - Symmetry test and change point detection can be formulated as two-sample test problem;
  - has many real applications in many areas, such as politics, medicine, etc.
- Related Methods
  - T-test or F-test
  - Kernel-based two-sample test
  - Kolmogorov-Smirnov test
  - Mutual information based test

# Copula Entropy: Two-Sample Test

- The proposed statistic

$$T_{ce} = H_c(\mathrm{X}, Y_0) - H_c(\mathrm{X}, Y_1), \tag{23}$$

where $\mathrm{X} = (X_1, X_2)$ is for two samples $X_1 = \{X_{11}, \cdots, X_{1m}\}$ and $X_2 = \{X_{21}, \cdots, X_{2n}\}$, and $Y_1 = (0_1, \cdots, 0_m, 1_1, \cdots, 1_n)$ and $Y_0 = (1_1, \cdots, 1_{m+n})$ are the labels for the null and the alternative hypothesis.

  - non-parametric multivariate two-sample test
  - defined as the difference between the copula entropies of the null and the alternative hypothesis;
  - $T_{ce}$ is small if $H_0$ is true.

- The estimator
  - estimating the two terms in (23);
  - calculating the estimated statistic.

# Copula Entropy: Two-Sample Test

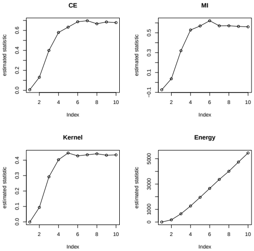
Simulation Experiments

- Data
  - bivariate normal distribution with different means
  - bivariate normal distribution with different variances
  - bivariate Gaussian copula with normal and exponential marginals
- Compared methods
  - Kernel-based test
  - Energy distance-based test
  - Mutual information-based test

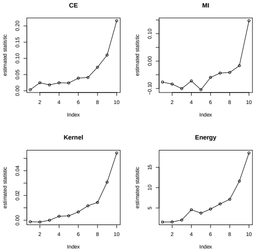
# Copula Entropy: Two-Sample Test

## Simulation Results

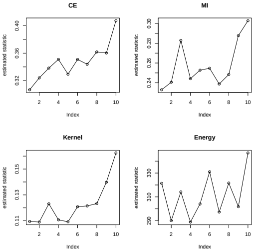
- Bivariate normal distribution with different means




- Bivariate normal distribution with different variances




- Bivariate normal copula with different variances



# Summary

- The theory of Copula Entropy was developed from copula theory, and a non-parametric method for estimating CE was proposed.
- CE was proposed to test statistical independence and conditional independence (transfer entropy).
- CE was applied to solve 8 fundamental statistical problems, including association discovery, structure learning, variable selection, causal discovery, time lag estimation, system identification, multivariate normality test, and two-sample test.

# References


1. Jian Ma and Zengqi Sun. “Mutual information is copula entropy”. In: *Tsinghua Science & Technology* 16.1 (2011). See also arXiv preprint arXiv:0808.0845 (2008), pp. 51–54
2. Jian Ma. “Discovering Association with Copula Entropy”. In: *arXiv preprint arXiv:1907.12268* (2019)
3. Jian Ma and Zengqi Sun. “Dependence structure estimation via copula”. In: *arXiv preprint arXiv:0804.4451* (2008)
4. Jian Ma. “Variable Selection with Copula Entropy”. In: *Chinese Journal of Applied Probability and Statistics* 37.4 (2021), pp. 405–420
5. Jian Ma. “Estimating Transfer Entropy via Copula Entropy”. In: *arXiv preprint arXiv:1910.04375* (2019)
6. Jian Ma. “Multivariate Normality Test with Copula Entropy”. In: *arXiv preprint arXiv:2206.05956* (2022)
7. Jian Ma. “Identifying Time Lag in Dynamical Systems with Copula Entropy based Transfer Entropy”. In: *arXiv preprint arXiv:2301.06037* (2023)
8. Jian Ma. “System Identification with Copula Entropy”. In: *arXiv preprint arXiv:2304.12922* (2023)
9. Jian Ma. “Two-Sample Test with Copula Entropy”. In: *arXiv preprint arXiv:2307.07247* (2023)
10. Jian Ma. “copent: Estimating Copula Entropy and Transfer Entropy in R”. In: *arXiv preprint arXiv:2005.14025* (2020)


http://arxiv.org/a/ma_j_3

# Softwares

- Official

  The **copent**[14] package in R and Python for estimating copula entropy, transfer entropy, and the statistic for multivaritate normality test are available on CRAN and PyPI respectively. The source codes are provided on GitHub.

  `https://cran.r-project.org/package=copent`

  `https://pypi.org/project/copent/`

  `https://github.com/majianthu`

- Third-Party

  The third-party implementations of the CE estimator include the **cylcop** package in R, the **MLFinLab** package in Python, the **CopEnt.jl** package and the **CausalityTools.jl** package in Julia, and the **gcmi** package in Matlab and Python.

14 Jian Ma. "copent: Estimating Copula Entropy and Transfer Entropy in R". In: *arXiv preprint arXiv:2005.14025* (2020).

# My Golf


Enjoy the Power of Copula Entropy!